A Number Partitioning Approach to Rhythm and its Application to Analysis of Ligeti's *Musica Ricercata 4*

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Abstract: We present a model for rhythm organization based on partitions of integer numbers. Firstly, we introduce a simple notation for coding rhythm patterns in terms of partitions. With this we present an analysis of rhythm patterns in Ligeti's Musica Ricercata IV for Piano. We show that, with few exceptions, Ligeti used the partitions of number 6 to get rhythm variations on the right hand against a balanced ostinato on the left hand. In addition, we show the usefulness of the so called Hasse Diagram, which can be used as a formal pre-compositional device for rhythm patterns.

Keywords: Musica Ricercata. Rhythm Patterns. Number Partitioning. Class of Equivalence.

Título: Uma abordagem de partição de número para ritmo e sua aplicação à análise de *Musica Ricercata 4* de Ligeti

Resumo: Apresentamos um modelo de organização rítmica baseado em partições de números inteiros. Indicamos uma notação simples para a codificação de padrões de ritmo em termos de partições e as aplicamos em uma análise dos padrões rítmicos na *Musica Ricercata IV*, para piano, de Ligeti. Mostramos que, com poucas exceções, Ligeti usou as partições do número 6 para obter variações de ritmo na mão direita contra um ostinato equilibrado na mão esquerda. Além disso, mostramos a utilidade do chamado Diagrama de Hasse, que pode ser usado como um dispositivo formal pré-composicional para padrões rítmicos.

Palavras-chave: *Musica Ricercata*. Padrões rítmicos. Partição de números. Classe de equivalência.

1. Introduction

Around the early 1950s, a period of great political repression in Hungary under the pro Soviet regime, Ligeti composed music which had no opportunity to be performed due to strong censorship. So, the fate of his *Musica Ricercata* (MR) was a "bottom drawer" (Steinitz, 2003). This set of eleven short pieces for piano contains a "minimalist program" of composition, with an increasing number of allowed pitches for each piece culminating with the last one in an idiosyncratic" twelve-tone type" approach even before Ligeti had any acquaintance with their protagonists. Recently, D. Grantham (Grantham, 2014) made an extensive study of macrostructures of all MR movements, mainly through a descriptive analysis. In this work we are most interested in a finer structure of Rhythm Patterns of Musica Ricercata 4, more specifically, on the right hand rhythm figurations against the almost ubiquitous rhythm pattern of the left hand ostinato shown in Figure1. Ligeti himself pointed out the structural importance of rhythm construction in all movements of Musica Ricercata (Steinitz, 2003).

Fig. 1 – Ostinato Cell: a balance between F♯ − A and G − B♭.

In this short piece, Ligeti uses a chromatic selection of five pitches, namely {F♯, G, G♯, A, B♭}.

2. Partitioning and Rhythmic Coding of Musica Ricercata IV

Consciously or not, he also used a number of rhythm patterns which can be classified, up to some few exceptions, as partitions of the number 6. *Partitions* of a natural number n, which we denote by (n) is the set of all ways (order not include) to write n as a sum of positive integers (Andrews, 1976). A simple example is given by

$\Phi(4) = \{1+1+1+1, 1+1+2, 1+3, 2+2, 4\}$. The number of elements of $\Phi(n)$ we denote $P(n)$. So, $P(4) = 5$. Applications of partitions of natural numbers to musical analysis and composition were extensively studied by Gentil-Nunes (2010). For $n = 6$ the number of partitions is $P(6) = 11$, which is easy to check. Now, the number 6 is the best choice in order to apply Theory of Partitions to the rhythm space Ligeti explores in this piece. This is because our analysis takes into account bars and beats in a fixed time signature. Musically we use the idea of partition of a natural number in a particular way: we code rhythm figures, notes and rests in units of eighth note, as follows:

$$♪ = 1, ♩ = 2, ♩. = 3, 𝅗𝅥 = 4$$

and so on. The correspondent rests have the same value with negative sign. A whole note has value 8 and a sixteenth note is represented by the fractional value 1/2 and its correspondent rest by -1/2, and so on. However, these fraction values are exceptions in our approach using partitions of integer numbers.

From this coding it is easy to see the right hand bars, from bar 1 to 60, can be coded as small lists of integer numbers with the duration values as above. Each small list corresponds to a bar, since we are interested in the rhythm variety of the right hand against that one of the left hand ostinato. The following section, bars 61-97, is just a repetition of bars 2 to 37 and, therefore, isn't necessary for our analysis. While the rhythmic ostinato cell of the left hand is a balanced pattern l2 2 2l, the right hand sequence of list is more varied:

$$
\begin{array}{r}
-6 \mid -6 \mid -6 \mid -4 \mid -6 \mid 6 \mid 3\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1 \mid \\
\mid 1\,1\,1\,1\,2 \mid 1\,1\,4 \mid -2\,4 \mid 3\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1\,1\,1 \mid 1\,1\,1\,1 \mid \\
\mid 1\,1\,1\,1\,1\,1 \mid 3\,1\,2 \mid 2\,-2\,2 \mid 4\,2 \mid -2\,3\,1 \mid 4\,2 \mid -2\,3\,1 \mid \\
\mid 3\,1\,1\,3/4\,1/4 \mid 4\,2 \mid -6 \mid -6 \mid -4 \mid -4 \mid \\
\mid -6 \mid -4 \mid 1\,1\,4 \mid 6 \mid 6 \mid -2\,2\,2 \mid 6 \mid -2\,1\,-1\,-1\,1 \mid 6 \mid -2\,2\,2 \mid 6 \mid \\
\mid -2\,1\,-1\,-1\,1 \mid 4\,2 \mid -4 \mid -4 \mid 1\,-1\,-1\,1\,2 \mid 1\,1\,1\,-1\,-1\,1 \mid \\
\mid 1\,-1\,2\,2 \mid 6 \mid -2\,1\,-1\,-1\,1 \mid 4\,2 \mid -2\,2\,2 \mid 6 \mid 6 \mid 2\,2\,2 \mid -2\,2\,2 \mid 4\,2 \mid 4\,2 \mid
\end{array}
$$

Fig. 2 – Rhythm Coding of Musica Ricercata's Right Hand

Observe that, due to the time signature, our choice of the eight note as time unit, implies $n = 6$ for each bar with few exceptions. If we choose a quarter note, we must take $n = 3$, but this implies too many fractions in the representation and thus partitions don't apply anymore. On the other hand taking a sixteenth note as time unit implies $n = 12$ and $P(12) = 77$ whose correspondent set of rhythm patterns is much larger than necessary for the analysis of a rhythmically simple piece as MR4 is. So we think $n = 6$ is the optimal choice for MR4. In our analysis the calculations of partitions do not take into account the difference between note and rest, that is, partitions mean only the division of a measure in time intervals using an integer number of time units. Of course, most works don't fit entirely this requirement and, as mentioned above, some fractional numbers appear as we can see in Figure 2. It is possible to circumvent this problem by using a shorter time unit but this implies a bigger number to be partitioned and thus the number of partitions becomes greater, making rhythm analysis more complex and, in some cases, leading to no meaningful information. So, we consider these cases of fractional numbers as exceptions, as there are only a few of them.

3. A Partitioning Based Rhythm Analysis of MR 4

As is the case in the previous movements of Musica Ricercata, it is the superposition of rapid changing rhythms in the right hand against an *ostinato* in the left hand that makes its construction interesting, even using just five pitch classes. Observe that, due to most bars having a time signature of 3/4, the sum of the durations of figures within them, including notes and rests, is 6, according to our coding. Ligeti got great rhythm variety by taking different combinations of notes and rests whose durations sum up 6 as shown in Figure 2. In fact, in MR4 Ligeti used 8 partitions out of 11 possible, namely

{1 1 1 1 1 1}, {2 1 1 1 1}, {4 1 1}, {2 2 2}, {3 2 1}, {4 2}, {3 1 1 1}, {6}

Those not appearing in the score are: {2 2 1 1}, {3 3}, {5 1}.

Fig. 3 – Bars 41 to 44 with code lists |6 |–222 | 6 | –2 1 –1 –1 1| for the right hand.

Negative values in the lists denote rest durations, and taking this into account the real number of possibilities of rhythm patterns can be greater. For example, if in a partition $n = a_1 + a_2 + \cdots + a_k$ we allows each element to be note (positive value) or rest (negative value) the number of possible permutations (rhythm patterns) is, a priori, $p = 2^k$. Of course, due to element repetitions in some partitions, the rhythm patterns can be less than p.

On the other hand, in contrast with rhythm patterns, partitions are non-ordered sets of numbers. It's easy to see that each partition above has a different number of possible associated rhythm patterns, obtained just making permutations between numbers. Formally, each partition is a class of equivalence of rhythm patterns under permutation operation. For example, if we have a partition of a number $n = a_1 + a_2 + \cdots + a_k$ we can make $k!$ permutations of correspondent rhythms associated with it. Other combinatorial operations can be done on the partition and new rhythms can be generated. Of course, Ligeti made use of just a small set of possible rhythm patterns which are representatives of classes of equivalence. Interesting works can be constructed using even just one class of partitions as, for example, Steve Reich's *Clapping Music*: he uses just (restricted) cyclical permutations of the rhythm vector. In binary code it reads [1 1 1 0 1 1 0 1 0 1 1 0] where 1 means a note and 0 a rest. It can be rewritten as [3 2 1 2], just counting the consecutive notes (there are no consecutive rests), adding up to 8. In Clapping Music Reich uses just 12 out of $P(8) = 22$ possible partitions. In terms of composition Reich uses an explicit algorithm (left shifts of the rhythm pattern) and Ligeti, as much as we know, none.

Now, in rhythm analysis an important question is about the distribution and hierarchy of the rhythm patterns along a piece. Our approach in this work, in order to meet this requirement, consists of introducing a kind of taxonomy which is naturally attained by defining an order in the set of rhythm patterns here represented by their correspondent partitions. So the order (taxonomy) is defined for partitions of a number. This can be gotten through the so-called *Hasse Diagram* of partitions of a number (Zhao, 2008). The following definition provides a partial order on partitions.

Partial Order on Partitions (Zhao) Suppose that $x = (x_1, x_2, \dots, x_r)$ and $y = (y_1, y_2, \dots, y_s)$ are partitions of *n*. Then x dominates y, written $x \trianglerighteq y$ if

$$x_1 + x_2 \dots + x_i \geq y_1 + y_2 \dots + y_i$$

for all $i \geq 1$. If $i > r$ (respectively, $i > s$), then we take x_i (respectively, y_i) to be zero. This partial order is also named *Dominance Order*.

As an example, by the above definition, we have $\{3\ 3\} \trianglerighteq \{3,2,1\}$. Nevertheless, it's not possible to compare $\{3\ 3\}$ with $\{4\ 1\ 1\}$ since the inequality fails in both cases of comparison. In this case, we say the partitions are independent of each other. The *Hasse Diagram* shows the dominance and independence of all partitions as shown in Figure 4 for the case of number $n = 6$ which we use for the analysis of MR4. The dominance is represented by an arrow. Observe that the Hasse Diagram for our case of note (or rest) durations is obtained through augmentation from bottom to top. On the other hand, from top to bottom we have a fragmentation of rhythm patterns.

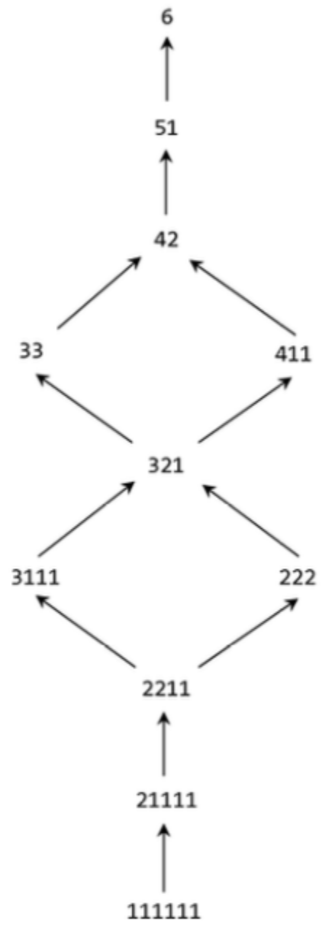

Fig. 4 – *Hasse Diagram* for number 6 (Gentil-Nunes 2009).

Following the sequence of partitions for MR4 (as shown in in Figure 2) through Hasse Diagram we find they go through three different sections of the diagram: the first one, at the very outset, its rhythm pattern corresponds to the highest element and suddenly moving to the least one. The middle section has great variation of rhythm patterns, which corresponds to a quasi-chaotic path through the central part of the diagram. In the third and last section rhythm patterns concentrate partially at the bottom and middle parts of the diagram but return to its highest parts and ending with partitions {4 2}. So the rhythm patterns show a kind of ABCA' form. Finally, observe that there are 13 out of 60 bars with full rest. This makes the left hand Ostinato be placed in the forefront for the listener.

4. Some Examples of Rhythm Pattern Generation

In this section we show a simple example how the above method can be used as a compositional tool to generate rhythm patterns. Firstly, observe, as mentioned above, that Ligeti didn't use all the partitions of 6 in MR4. In fact, a composition using partitions of a large number is likely to explore just a small subset of the partitions. In order to fix such a subset, the composer can use constraints of different kinds, such as symmetry, or even random choices. This economy in terms of rhythm patterns implies, in general, more complexity for other musical parameters. In this case the model of rhythm partitions is of little use. Its power is most revealed when the composer intends to use a bunch of varied rhythms. Below we present some examples of rhythm patterns generated with concatenation and superposition of partitions.

Consider $n = 8$. The number of partitions is $P(8) = 22$ and its Hasse Diagram is shown in Figure 5. Observe that from the bottom to the top, as well from right to left, the nodes growth by augmentation. Since partitions are non-ordered sets, and rhythm is an ordered set of durations we are free to include permutations on any node (rhythm partition) as well as accentuation. In addition, as we did for Music Ricercata 4 in Figure 2, we can include rests taking negative values in any partition and its permutations.

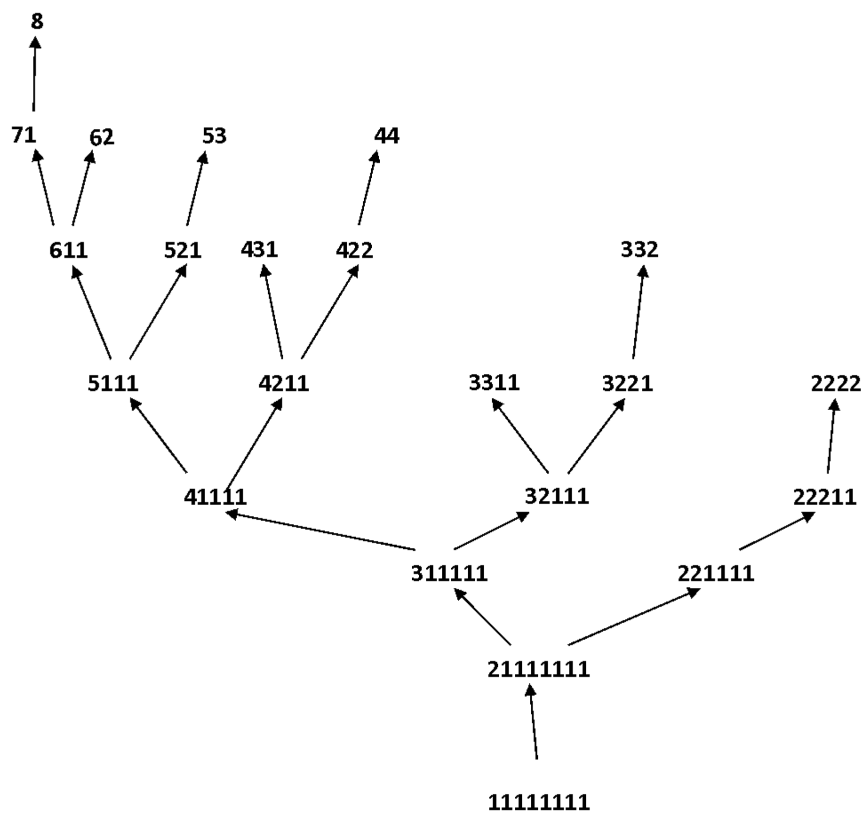

Fig. 5 – Hasse Diagram for Partitions of 8

Figure 6 shows an example of concatenation of partitions at the extreme limits of Hasse Diagram, namely: [11111111] in the first two bars and [8] in the third one. We've taken an eighteenth note as a time unit.

Fig. 6 – Using extreme partitions [11111111] and [8] at the edge of Hasse Diagram.

Figure 7 shows an example using permutations of partitions from two branches of the Hasse Diagram where the values in a partition count the distance, in time units (in this example, an eighteen note), between chords.

Fig. 7 - Polyrhythm using different branches of Hasse Diagram. The last two Viola and Cello's bars are permutations of the first one.

Many other examples can be constructed using the method of time organization above for musical structures in a score. However, observe that for large numbers the set of partitions has too many elements and so there is no point to analyze works with small rhythm variability.

5. Conclusions and Perspectives

We presented above an extension of Gentil-Nunes approach to analysis and composition based on Partitions of an integer number. Complementing Gentil-Nunes approach which uses partitions exploring a set of notes we've applied to the horizontal dimension of time, that is, rhythm. That approach can be useful in order to search patterns from a score. In fact, the method can be applied to search patterns in time organization of musical structures. However, it is important to stress that our method may not represent the actual rhythm on the score since in our analysis, for example, some artificial breaks are needed to accommodate partitions, as showed in our analysis of Ligeti's Musica Ricercata 4. Nevertheless, this can be thought of as a second order rhythm organization. On the other hand, it's possible to generalize the above method just measuring distance, in time units, between structures. These could be vertical such as notes, chords, clusters; horizontal such as tuplets, melodic patterns, or even between any kind of blocks with arbitrary vertical and horizontal extensions. In addition, it is also possible to extend the method by defining distance between accentuations and techniques applied on structures when appropriated.

The simple examples above show the joint studies of partitions of numbers and associate classes of equivalence under permutations can be a useful tool for analysis and composition of rhythm patterns in many other works of music.

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